Mechanics (PHYC 190)   Orbital Quadrature

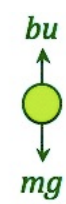
Purpose

A mathematical technique called quadrature will be employed to simulate the orbit of a planet around a central gravitational force. Experimentation will then proceed on the mathematical model in place of actual celestial objects.

Background

Newton's Law of Gravitation constrains the possible shapes of orbits. Unfortunately, real objects in orbits tend to require far too long an observational period to work as a single, practical laboratory activity. Instead, it is possible to explore orbits using ***quadrature*** and computer simulation.

Quadrature is the iterative approximation of an integral-based mathematical equation using small intervals of the independent variable(s) to compute changes in the dependent variable(s).[1] For example, the Free Body Diagram of an object of mass *m* falling while undergoing linear drag is shown at right. Using Newton's Second Law and summing the forces in the downward direction yields $mg - bu = ma$. Since the acceleration *a* is the derivative of the speed *u*, this can be written as $\dfrac{du}{dt} = g - \dfrac{b}{m}u$.



While it is possible to solve this equation for the speed explicitly, it is far easier to replace the derivatives with deltas and find the change in *u* for finite time increments: $\Delta u = \left[g - \frac{b}{m}u\right]\Delta t$. Once this value is found, the next iteration of *u* is the last value plus Δu, and the result is then used in computing yet another iteration. As long as the fixed value for Δt is very small, the results will be a good approximation to the true function.

Suppose an object of mass *m* is in orbit about a central mass m_0 located at the origin. The gravitation force vector depends on the position: $\vec{F} = -\hat{r}Gmm_0 / r^2$, where $\hat{r} = \hat{x}\frac{x}{r} + \hat{y}\frac{y}{r}$, and $r = \sqrt{x^2 + y^2}$. According to Newton's 2nd Law, the acceleration of this mass will be $\vec{a} = \vec{F}/m = -\hat{r}Gm_0 / r^2$. The change in velocity is therefore the time integral of this acceleration: $\Delta\vec{u} = \int \vec{a}\,dt = -\int \dfrac{\hat{r}Gm_0}{r^2}dt$. To evaluate this by quadrature, the integral will be replaced by the function multiplied by a small time interval Δt. This will be much easier to understand, however, if the *x*- and *y*-components are dealt with separately.

Using the definition of *r-hat* above, the *x*-component of acceleration is $a_x = -Gm_0 x / r^3$. The *x*-component of the change in velocity is therefore $\Delta u_x = -\int \dfrac{Gm_0 x}{r^3}dt$, which becomes $\Delta u_x = Gm_0 x \Delta t / r^3$.

The delta on the left in the final equation above is $u_x - u_{0x}$ by definition, where u_{0x} is the x-component of velocity from the previous iteration; therefore $u_x = -Gm_0 x \Delta t / r^3 + u_{0x}$. This is needed to find the x-component of position since displacement Δx is the time integral of u_x. Once again, the integral will be replace with simple multiplication by the time interval Δt. To avoid confusion between the two instances of x that will appear, all prior-iteration values from the equation will receive a subscript zero: $u_x = -Gm_0 x_0 \Delta t / r_0^3 + u_{0x}$. The result of multiplying this by delta-t again is $\Delta x = -Gm_0 x_0 \Delta t^2 / r_0^3 + u_{0x} \Delta t$. This is the incremental change in x during each iteration. The incremental change in y is of the corresponding form.

Looking at this formula, a number of parameters are needed to begin computing iterations of x and y. Specifically, constant values must be chosen for m_0 and Δt, and starting values are needed for x_0, y_0, u_{0x}, and u_{0y}. Assuming all of these are known, one iteration of the quadrature consists of the following steps. First, r_0 is found from x_0 and y_0. Second, x and y are found from x_0, y_0, r_0, u_{0x}, and u_{0y}. Third, u_x and u_y are found from x_0, y_0, r_0, u_{0x}, and u_{0y}. Last, to prepare for the new iteration, x is now called x_0, y becomes y_0, u_x becomes u_{0x}, and u_y becomes u_{0y}.

Procedure

Programming languages take a long time to learn, but the quadrature needed for this lab can be performed using a much friendlier tool: Microsoft Excel.[2] Find this program in the main applications menu and open it. When it inquires, select "Blank Excel Workbook."

Excel is a spreadsheet application, so the window will show a white sheet with black gridlines after it finishes loading. The grid will be composed of rectangular *cells*. Data or equations can be entered into these cells, and each cell has a unique *address* labeled by the letter at the top of its column and the number at the left of its row. The upper-left cell, for example, is *A1*.

1. For simplicity, the starting orbital radius of the planet being modeled here is 1 unit. Click in cell *A1* and type `1` followed by *return*. This is the starting x coordinate of the planet. The starting y coordinate will be in cell *B1*, and its value will be `0`.

2. The planet needs starting velocity components as well or it will simply fall straight into the sun. The planet is starting out directly to the right of the sun on the standard xy-plane, so it needs to move in the y direction to attempt an orbit. Enter `0` into cell *C1* and `1` into cell *D1*. These are u_{0x} and u_{0y}, respectively.

3. It will be necessary to know the planet's current distance from the origin for the next iteration (which will happen on the next row). Cell *E1* will therefore contain the first actual calculation of this spreadsheet. The formula is $r_0 = \sqrt{x_0^2 + y_0^2}$, but Excel would have no idea what these variables are. Instead, it must be given the cell addresses where they can be found (*A1* and *B1*). In cell *E1*, enter `=sqrt(A1^2+B1^2)` and be sure to include the equals sign so that Excel knows it is supposed to do a calculation rather than just display the text of the formula. A number will appear in the cell if the equation was entered correctly. If it was not, simply click on the cell again and reenter the formula.

4. It's now time to start the iteration process. The second row of five columns will all be calculations based on the quadrature formulas derived in the Background section above. For simplicity in this assignment, the universe in which this planet exists has a universal gravitational constant of 1, and the star's mass will also be 1. The time increment Δt doesn't need to be exceedingly small: 0.01 will suffice for this simulation. The new value for x is therefore $x = -x_0 0.01^2 / r_0^3 + u_{0x} 0.01 + x_0$. Click on cell *A2* and then enter the formula `=-A1*0.01^2/E1^3+C1*0.01+A1` to compute the new x-location. (The asterisks are there to indicate multiplication and Excel becomes confused without them.) The formula for the new y-location in cell *B2* is `=-B1*0.01^2/E1^3+D1*0.01+B1`.

5. New velocity components will be needed for the next iteration. In cell *C2*, enter `=-A1*0.01/E1^3+C1` and in cell *D2* enter `=-B1*0.01/E1^3+D1`.

6. Cell *E2* needs to contain the new orbital radius. Rather than retyping the formula used in cell *E1* with the new values of *A2* and *B2*, it is much more convenient to take advantage of Excel's design: Click on cell *E1*, then move the cursor up to the *Edit* menu at the top of the screen, and click and drag down to *Copy*, and release the mouse button. Now click on cell *E2*, then click and drag on the *Edit* menu down to *Paste* and release. Excel will automatically fill cell *E2* with a location-adjusted copy of the old formula. A new number should appear there and now all five cells of row 2 on the page will have numbers in them if everything has gone right so far.

7. Fully one thousand rows of values are needed for this simulation. Even doing this by copying and pasting would take hours. Instead, there is a much easier way: Click and drag from somewhere in cell *A2* to somewhere in cell *E2* to highlight the row of values (and their hidden formulae), and release the mouse button. Now move the cursor over to the bottom right corner of cell *E2* and the white plus sign cursor should turn black. While it is black, click and drag straight down, off the bottom of the page, and Excel will start scrolling down the sheet. Don't release the mouse button until row 1000 is reached. A dashed box will be visible to guide this process. The sheet should now contain a block of numbers five columns wide and 1000 columns tall.

8. It is now time to graph the coordinates. Go back to the top of the sheet using the scroll bar on the right side of the screen, and find the letter *A* at the top of the first column. Click and drag from somewhere in that box to somewhere in the box beside it containing *B*, then release. This will select the *x*- and *y*-values from the entire sheet. Now go up to the menu bar at the top of the screen and click and drag on *Insert* down to *Chart*, and release. Just above the column letters, a list of chart types will appear. Click *X Y (Scatter)* and then *Marked Scatter* and a graph should appear on the sheet somewhere.

9. The graph isn't set up appropriately for this project. It needs to be resized and rescaled. Click and drag the very lower-right corner of the graph to reshape it into a square instead of a rectangle. Once this is done, click on the words *Series 1* inside the chart and then hit the delete key to remove the chart legend. Now double-click on any number in the vertical (*y*) axis to open up the axis formatting window and look for the word *Scale* on the left side of the window and click it. The result should be five text input areas for the chart's minimum, maximum, units, and axis-crossing. Click in the first of these, delete the number there, and replace it with -2. Click in the second, third, and fourth and replace them each with 2, and the last with 0. Each time, a checkbox will be automatically deselected, which will keep Excel from continually reshaping the chart each time the data values change, as this would be confusing. Now click *OK* in the lower right of the formatting window, and then do the same process with the horizontal axis.

Questions

The finished chart should show a circular orbit with a radius of 1 unit. The benefit of using Excel is that any change made to the starting values of x_0, y_0, u_{0x}, and u_{0y} in cells *A1*, *B1*, *C1*, and *D1* will automatically update the entire array of values and the plotted points on the chart. This will make it easy to explore orbits.

1. Change the value of u_{0y} in cell *D1* to 0.5 by clicking on that cell, typing the new value, and pressing return. Observe the new orbit. Explain why an object starting with a slower orbital speed from the same position would behave this way.

2. Change the value of u_{0y} to 1.15 and observe that the orbit appears incomplete. Explain what has caused this.

3. Change u_{0y} to 0.3 and observe the behavior. Is this a result of the simulation's limitations or could a planet actually do this? To answer the question, try a value of 0.2 instead and attempt to determine what appears to have happened to the planet. Remember that the planet started out in an orbit with a very limited amount of energy due to the very low speed.

4. Change u_{0x} in cell *C1* to -0.7 (instead of zero), and u_{0y} to 0.7 and observe the result. What is different this time from the orbit in Question 1 above?

5. Change u_{0x} to -0.5 and u_{0y} to 1.5 and observe that the “orbit” seems to get straighter as it moves away. Will the planet ever come back, and if not, why not? (Recall that orbits around central forces are conic sections, and there are other conic sections besides just circles and ellipses.)
6. One form of Kepler’s Third Law of planetary motion can be derived by creating a free-body diagram for a planet in a circular orbit due to gravitational attraction. Do this for the simulated solar system with G and m_0 set to 1. Use the results to predict the initial speed u_{0y} required to establish a circular orbit of radius 0.5. Test this prediction by setting x_0 in cell *A1* to 0.5, u_{0x} to 0, and u_{0y} to the predicted value and looking at the simulated results.
7. Based on the equation derived in Question 6, should the planet Mercury in its circular orbit close to the sun be expected to move faster than planet Neptune in its circular orbit very far from the sun, or should it instead be expected to move more slowly?
8. Set u_{0x} and u_{0y} to zero and observe the result. If a real planet in orbit around a real star stopped like this for some reason, could it possibly behave that way, and if not, why not?
9. Set x_0 to 1 and u_{0y} to -1 and observe the result. Consider the implications of this setting. Could a planet really do this, and if not, why not?
10. When in an elliptical orbit, a planet’s farthest distance from the sun is called *aphelion* (r_{max}) and its closest distance is called *perihelion* (r_{min}). The *eccentricity* (e) of the orbit is a number between zero and one that describes how elliptical it is, zero being circular.[3] It can be shown that $e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}$. Set u_{0y} to 0.6 and scroll down through the spreadsheet to find the x-value when the y-value switches from positive to negative. (Interpolate between the two values on either side of this change.) This result will give the negative of the perihelion distance. The aphelion distance is 1 since that is the value of x_0 when y_0 is zero. Use these *apsidal distances* to compute the eccentricity.

Report Tips

It is impractical to copy the entire numerical table from Excel into a lab report. To copy a chart from Excel, however, it is wisest to save it as a picture by right-clicking near the very top and a menu option will appear to save it in a format such as PNG. The saved picture, when copied and pasted into the document, will not change when further alterations are made in the Excel file.

References

[1]MathWorld, mathworld.wolfram.com/Quadrature.html.

[2]Computer Program, EXCEL, (Microsoft, Redmond, WA, 2011).

[3]H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*, 3rd ed. (Pearson, San Francisco, CA, 2002), p. 94.